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Semiclassical steady-state analysis of a degenerate two-photon laser

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Abstract. The steady-state analysis of a single-mode two-photon laser are treated semiclassically by using the Maxwell-Bloch equations. The theory is applied to a ring-laser model. We find similarities and significant differences between the one- and two-photon polarizations of the medium, population inversion and mode-pulling formula. The population inversion and the longitudinal variation of the steady-state modulus of the field are studied numerically.

PACS. 42.55.Ah General laser theory – 41.20.Jb Electromagnetic wave propagation – 42.25.Ja Polarization

1 Introduction

Recent work [1] in view of continuing technological improvements in microcavities even at optical frequencies has motivated the examination of certain aspects of the two-photon laser theory that are fundamental to the process. There aspects have their counterpart in the usual single-photon laser, but rather different behavior is to be expected in the two-photon case, owing to the essential non-linearity of the process even at weak signal. We have here in mind a degenerate two-photon laser with the atom pumped to the upper state connected to the lower one of the lasing transition by a two-photon process. Although not realized as yet in this pure form, it probably is a matter of short time before that occurs [2–4]. The situation here is somewhat different from the dressed states scheme that has already been demonstrated experimentally some time ago by Mossberg and collaborators [5,6].

The issue we have in mind has to do with the steady-state behavior of the system, taking into account the dependence on the relevant magnitudes such as the field strength and the inversion. This is most conveniently accomplished in a semiclassical formalism in terms of the Maxwell-Bloch equations. Related treatments based on either simple rate equations [7], discussing threshold conditions, or the Maxwell-Bloch equations without the spatial dependence, have been presented in the literature [8–10]. What we have studied and present below is essentially the generalization of the complete Maxwell-Bloch equations, usually employed in the single-photon laser theory, to the two-photon case. We have found it most convenient to use a formulation presented some time ago by Narducci in the semiclassical theory of the single-photon laser [11].

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2 Derivation of equations

We consider the coupled set of Maxwell-Bloch equations, in the usual rotating-wave approximation, which govern our two-level atom when the dipole forbidden transition is replaced by a two-photon one. We consider the degenerate case, in which pairs of photons with the same frequency are created or absorbed, and we analyze the stability of the steady state. We adopt a semiclassical laser model based on a microscopic two-level Hamiltonian. We assume a collection of identical homogeneously broadened two-level atoms, with energies E_1 and E_2 such that $E_2 > E_1$ with $E_2 - E_1 = \hbar\omega_{\mathrm{a}}$, ω_{a} the atomic transition frequency and a generated unidirectional single-mode classical electric field

$$E(z,t) = \frac{1}{2}\{E_0 e^{i(k_{\mathrm{c}}z-\omega_{\mathrm{c}}t)} + \mathrm{c.c.}\} \tag{1}$$

inside a ring cavity. Here E_0 is the real field amplitude, k_{c} the wave number, z the cavity axial direction and ω_{c} represents the unloaded cavity frequency. The atoms interact with the field in the dipole approximation via a two-photon transition, where these states are assumed to have the same parity, and thus are not connected by a one-photon transition.

Adopting the plane-wave approximation, we reduce the Maxwell-Bloch equations to

$$\frac{\partial \bar{F}}{\partial z} + \frac{1}{c}\frac{\partial \bar{F}}{\partial t} = -\alpha \bar{P}\bar{F}^*, \tag{2}$$

$$\frac{\partial \bar{P}}{\partial t} = -(\gamma_1 + i\delta_{\mathrm{ac}})\bar{P} - \gamma_1 \bar{F}^2 \bar{D}, \tag{3}$$

$$\frac{\partial \bar{D}}{\partial t} = \gamma_2 \left\{ \frac{1}{2}(\bar{P}\bar{F}^{*2} + \bar{P}^*\bar{F}^2) - \bar{D} + 1 \right\}, \qquad (4)$$

where $\bar{F}, \bar{P}$ and $\bar{D}$ are the normalized output field, two-photon polarization and population difference, respectively, ($\bar{F} = \sqrt{\mu^{(2)}/\hbar\gamma_1\gamma_2}\bar{E}_0$), $\mu^{(2)}$ the effective dipole matrix element for the two-photon transition, γ_1 and γ_2 are the decay rates of two-photon polarization and population difference, respectively. α denotes the unsaturated gain constant per unit length of the active medium ($\alpha = 2N\omega_c(\mu^{(2)})^{2/3}/2c\hbar\varepsilon_0\gamma_1$), where N is the number of atoms per unit volume, ε_0 the vacuum electric permeability and c the speed of light. We denote by $\delta_{ac} = \omega_a - 2\omega_c$ the detuning of the cavity mode from two-photon resonance, and the term proportional to $(\gamma_1 + i\delta_{ac})$ is similar to that of the one-photon case. Equations (2-4) are the same as those for one-photon two-level system with the following substitutions $\bar{F} \to \bar{F}^2$ and $\omega_c \to 2\omega_c$. Equations (2-4) here are non-linear in $\bar{F}$, as is also the case for the one-photon two-level system [11]. The major difference between the two cases is that the equations governing this system involve non-linearity of higher order. Equations (2-4) have been derived by assuming an effective Hamiltonian, *i.e.*, by assuming a pure two-photon interaction between the two-level atom and electromagnetic field. This approach neglects residual effects of any largely detuned one-photon transitions between the lasing levels and other atomic levels [12,13]. A more precise approach consists in assuming an exact or microscopic interaction Hamiltonian that describes the interaction of the electromagnetic field with a three-level cascade atomic scheme [14,15]. When the intermediate atomic level is far from one-photon resonance, the one-photon coherence can be adiabatically eliminated and the resulting two-photon laser equations are similar to the present equations but include three additional detuning terms describing frequency shifts.

The model is completed by appropriate boundary conditions which, in the case of a traveling wave ring-cavity resonator, take the form

$$\bar{F}(0,t) = R\bar{F}(L, t - (\Lambda - L)/c), \qquad (5)$$

where L is the length of the active medium; while the full length of the ring resonator is Λ, and R is the amplitude reflectivity of two of the mirrors. For simplicity, the remaining optical surfaces that are needed to complete the ring are assumed to be ideal reflectors.

3 Steady state

To study the steady state, we consider the equations in the long-time limit by setting the time derivatives equal to zero, for an active medium detuned by an arbitrary amount δ_{ac} from the resonant cavity mode. Under these conditions, the output field is expected to oscillate with a carrier frequency ω_L which is neither equal to ω_c nor $\omega_a/2$, but to some intermediate value determined by the cavity and atomic parameters. For this reason, we look for steady-state solutions of the type

$$\bar{F}(z,t) = \bar{F}_{st}(z)e^{-i\delta\omega t}, \qquad (6)$$

$$\bar{P}(z,t) = \bar{P}_{st}(z)e^{-i2\delta\omega t}, \qquad (7)$$

$$\bar{D}(z,t) = \bar{D}_{st}(z) \qquad (8)$$

where $\delta\omega$ is the frequency offset of the operating laser line from the resonant mode (*i.e.* $\delta\omega = \omega_L - \omega_c$). Of course, $\delta\omega$ is unknown and must be calculated. The atomic variables can be determined at once as functions of the stationary field profile:

$$\bar{P}_{st}(z) = -\bar{F}^2_{st}(z)\frac{1 - i\Delta}{1 + \Delta^2 + |\bar{F}_{st}(z)|^4}, \qquad (9)$$

$$\bar{D}_{st}(z) = \frac{1 + \Delta^2}{1 + \Delta^2 + |\bar{F}_{st}(z)|^4}, \qquad (10)$$

where the detuning parameter Δ is defined as $\Delta = (\delta_{ac} - 2\delta\omega)/\gamma_1$. Equations (9, 10) are similar to those for the one-photon two-level system with the following substitutions: $\bar{F}_{st} \to \bar{F}^2_{st}$ and $\delta\omega \to 2\delta\omega$. The steady-state polarization and the field envelope are generally out of phase from one another by an amount that depends on the detuning δ_{ac} and the position of the operating laser line. On resonance, however, $\bar{P}_{st}$ and $\bar{F}_{st}$ have the same phase. The steady-state population difference (inversion) saturates at high intensity levels in the sense that $\bar{D}_{st} \to 0$ as $|\bar{F}_{st}| \to \infty$. To determine the value of the output field and the form of its longitudinal profile in steady state, it is convenient to represent the field amplitude in terms of its modulus and phase (both space dependent):

$$\bar{F}_{st}(z) = \rho(z)e^{i\theta(z)}. \qquad (11)$$

Substituting equation (11) and equation (6) into equation (2), we have

$$\frac{d\rho}{dz} = \frac{\alpha\rho^3}{1 + \Delta^2 + \rho^4}, \qquad (12)$$

$$\frac{d\theta}{dz} = \frac{\delta\omega}{c} - \frac{\alpha\Delta\rho^2}{1 + \Delta^2 + \rho^4}. \qquad (13)$$

The two coupled equations can be combined to yield the first integral of the problem

$$\ln\frac{\rho(z)}{\rho(0)} = -\frac{1}{\Delta}\left[\theta(z) - \theta(0) - \frac{\delta\omega z}{c}\right], \qquad (14)$$

while equation (12) can be integrated at once to give

$$(1 + \Delta^2)\left(\frac{1}{\rho^2(z)} - \frac{1}{\rho^2(0)}\right) - \rho^2(z) + \rho^2(0) = -2\alpha z. \qquad (15)$$

The boundary conditions, expressed in terms of the field modules and phase provide the two constraining relations

$$\rho(0) = R\rho(L), \tag{16}$$

$$\theta(L) - \theta(0) = -\delta\omega(\Lambda - L)/c + 2\pi j, \tag{17}$$

where j is, *a priori*, equal to zero or any positive or negative integer. This implies that, in principle, the boundary conditions can be satisfied by more than one solution. This is not surprising in view of the resonant nature of the cavity. The result is important, however, because it suggests the possibility of coexisting steady states and mode-mode interactions. Then, the output laser intensity can be calculated at once from equation (15) after selecting $z = L$, and using the boundary condition (16), with the result

$$\rho^4(L) = \frac{2\alpha L}{1 - R^2}\rho^2(L) - \frac{1 + \Delta_j^2}{R^2}, \tag{18}$$

where $\Delta_j = (\delta_{\rm ac} - 2\delta\omega_j)/\gamma_1$, $\delta\omega_j$ is the operating laser frequency. Equation (18) has two roots and at laser threshold the intensity is not vanishing. There is coexistence of three solutions (above threshold): the trivial and two other solutions with intensity different from zero. One solution grows with the pumping parameter up to an asymptotic value for pumping going to infinity. The other solution decreases towards the zero solution as the pumping grows to infinity. This means that the threshold is not a second-order phase transition as in the case of single-photon lasers.

The quantity $c \mid \ln R \mid /\gamma_1 \Lambda$ represents the decay rate of the cavity field, $2\pi c/\Lambda$ is the spacing between adjacent cavity resonances. After introducing the abbreviations $K = c \mid \ln R \mid /\Lambda, \alpha_1 = 2\pi c/\Lambda$ and from equation (14), we obtain

$$\delta\omega_j = \omega_{\rm L} - \omega_{\rm c} = \frac{K\delta_{\rm ac} + \alpha_1\gamma_1 j}{\gamma_1 + 2K}, \tag{19}$$

where the sub-index j reminds us of the possible existence of multiple solutions. This is the well-known mode-pulling formula. It shows that the laser operating frequency is a weighted average of the atomic resonant frequency and the frequency of one of the cavity modes. The main difference between this result for two-photon laser and that for the one-photon laser [11] is a factor 2 in the denominator of equation (19) which makes the dependence of $\delta\omega_j$ on the parameters somewhat different.

4 Results of calculations

In order to keep the calculations presented in this paper as realistic as possible, we have chosen to apply our model for a real atomic system (for the transition $4p_{3/2}$-$6p_{3/2}$ in potassium). The reason for choosing this transition is the result of a compromise. On the one hand, one wants the energy of the photons involved to be as large as possible, and preferably in the optical regime. On the other hand, it is hard to find a two-photon transition in the optical regime with a large coupling, since a large two-photon coupling demands the existence of an almost resonant intermediate level with opposite parity. The transition mentioned above involves photons with an energy of $\simeq 7980$ cm^{-1}, *i.e.* near-infrared, and has a two-photon coupling that is orders of magnitude larger than the other candidates we looked at, due to the almost resonant $5s$ state. Besides the atom, we should also choose a cavity. In the model presented in this paper, we are assuming that only one mode of the cavity field is excited. For this to be true, the cavity should be rather small, since it then supports fewer modes, and these will be better separated in energy. Another advantage of having a small cavity is that the two-photon coupling $\mu^{(2)}$ will be larger, since it is proportional to V^{-1} (following the notation of Loudon) [16], V being the cavity volume. We have chosen the cavity volume $V = 10^{-15}$ m^3.

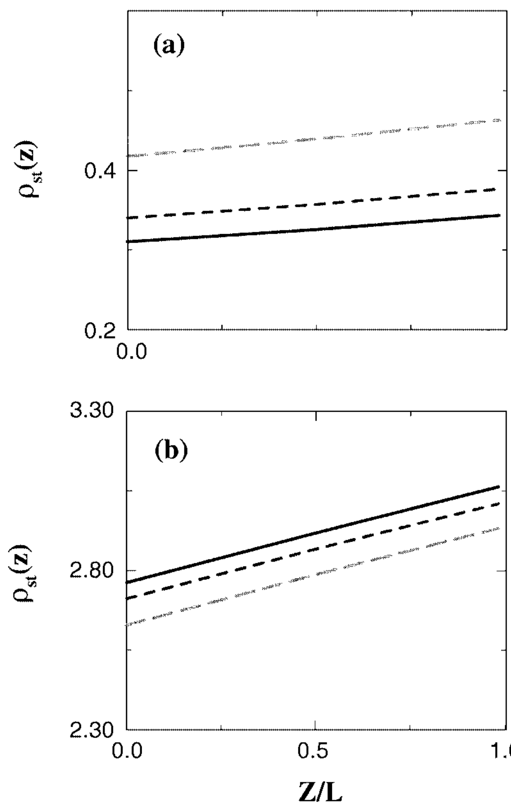

Fig. 1. The longitudinal variation of the steady-state modulus of the field $\rho_{\rm st}(z)$ as a function of z/L for $\alpha L = 1$, $R = 0.9$, $\delta_{\rm ac}/\gamma_1 = 0$ (solid line), $\delta_{\rm ac}/\gamma_1 = 0.5$ (dotted line), and $\delta_{\rm ac}/\gamma_1 = 1$ (dashed line), (a) for the two-photon case and (b) for the one-photon case.

It is interesting to inquire into the longitudinal profile of the field under given steady-state conditions. This can be done using equation (15) with the boundary condition (16). The solution of this transcendental equation can be obtained easily by numerical means. Selected sample solutions are shown in Figure 1, where the longitudinal variation of the steady-state modulus of the field $\rho_{\rm st}(z)$ is plotted against z/L for an amplitude reflectivity of the mirrors $R = 0.9$, $\alpha L = 1$, $j = 0$, and for different values

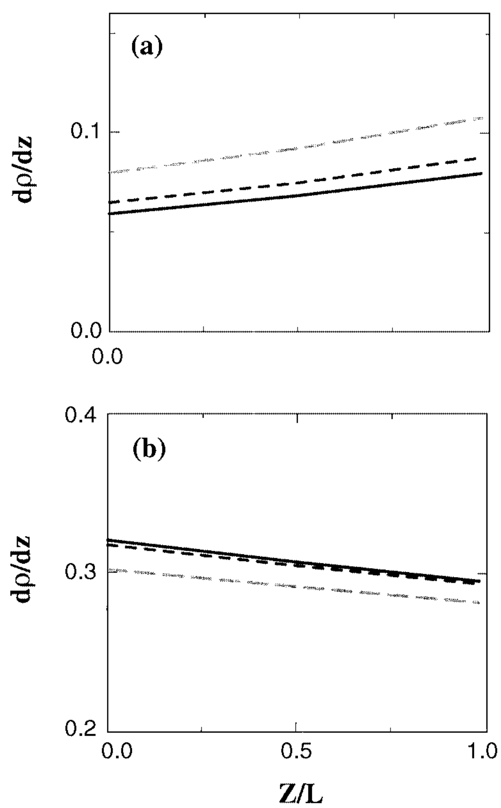

Fig. 2. The spatial derivative of the steady-state modulus of the field $\mathrm{d}\rho(z)/\mathrm{d}z$ as a function of z/L for $\alpha L = 1$, $R = 0.9$ and $\delta_{\rm ac}/\gamma_1 = 0$ (solid line), $\delta_{\rm ac}/\gamma_1 = 0.5$ (dotted line) and $\delta_{\rm ac}/\gamma_1 = 1$ (dashed line), (a) for the two-photon case and (b) for the one-photon case.

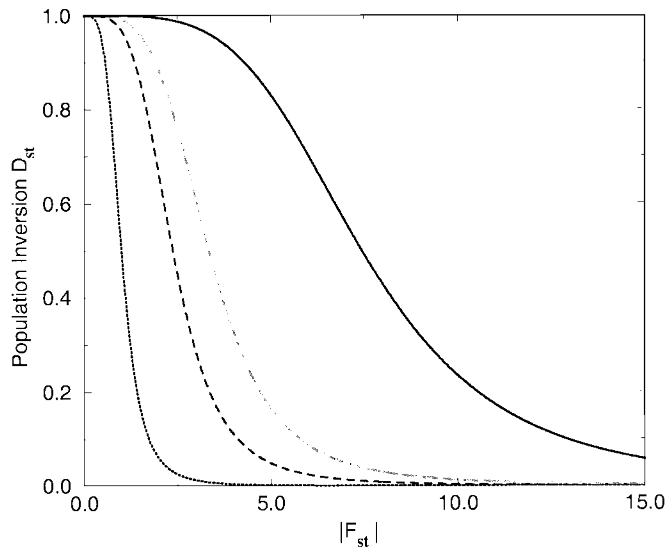

Fig. 3. The steady-state population inversion is plotted against the stationary field $\mid F_{\rm st} \mid$ for $\alpha L = 1$, $R = 0.9$ and $\delta_{\rm ac}/\gamma_1 = 0$ (dotted line), $\delta_{\rm ac}/\gamma_1 = 5$ (dashed line), $\delta_{\rm ac}/\gamma_1 = 10$ (long-dashed line), and $\delta_{\rm ac}/\gamma_1 = 50$ (solid line).

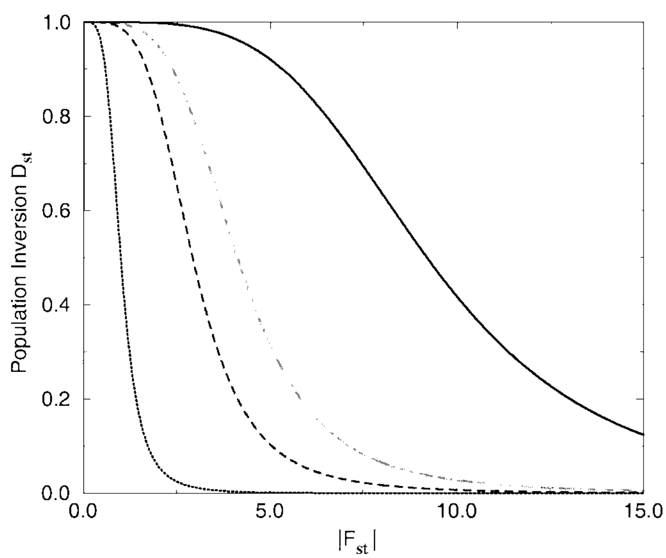

Fig. 4. The steady-state population inversion is plotted against the stationary field $\mid F_{\rm st} \mid$ for $\alpha L = 1$, $R = 0.5$ and $\delta_{\rm ac}/\gamma_1 = 0$ (dotted line), $\delta_{\rm ac}/\gamma_1 = 5$ (dashed line), $\delta_{\rm ac}/\gamma_1 = 10$ (long-dashed line), and $\delta_{\rm ac}/\gamma_1 = 50$ (solid line).

of the detuning parameters $\delta_{\rm ac}/\gamma_1$. In this case the field longitudinal profile undergoes a fairly large variation. The details of the spatial variation of the laser intensity inside the medium are governed by the degree of saturation and by the mirror reflectivity. The longitudinal variation of the steady-state modulus of the field for small z is smaller than it would be in the single-photon case, but the increase for large z appears to be faster, reflecting the additional non-linearity of the system. This is seen by considering the spatial derivative of $\rho(z)$ shown in Figure 2a. It is increasing with z, while the corresponding quantity for the single-photon case Figure 2b actually decreases for the same interval and the same parameters. Note that the results of Figures 1 and 2 are insenstive to the sign of $\delta_{\rm ac}$ since we have chosen $j = 0$ and in this case only $\delta_{\rm ac}^2$ appears in equations (15) and (16).

In Figure 3 the steady-state inversion is plotted against the field amplitude, for $R = 0.9$, $\alpha L = 1$, $j = 0$, and for different values of the detuning parameters $\delta_{\rm ac}/\gamma_1$. We show that saturation will occur at high intensity levels with increasing the detuning parameter $\delta_{\rm ac}/\gamma_1$. Also, with decreasing the amplitude reflectivity R, the steady-state population difference saturates at high intensity levels as seen in Figure 4. Also, from Figure 4 we see that the cavity quality (*i.e.* the reflectivity R becoming smaller) has a dramatic influence on saturation. Two-photon lasers and usual one-photon lasers are *a priori* very different systems since the former are based in an intrinsic nonlinear process, the two-photon stimulated emission, which depends on the field intensity. It is thus not surprising that the longitudinal variation of the steady-state modulus of the field $\rho_{\rm st}(z)$ of the two-photon lasers be very different from those of one-photon lasers. The most salient distinctive feature of the two-photon lasers is: the laser-off solution is always stable (thus implying the necessity of triggering for laser action). Moreover, self-pulsing emission is still possible in autonomous class-B two-photon lasers [13] (lasers for which the polarization decay rate largely exceeds the population and photon decay rates and on which no external modulation is exerted), a behavior that is in contrast with most laser models. Although the stability and dynamical properties of the two-photon lasers have been the subject of several studies [13–15,17–20], there are still many questions about the behavior of two-photon lasers that need to be addressed including the laser linewidth, coherence properties, instabilities in the output power, photon fluctuation noise, photon correlations and detailed studies of the threshold behavior. These studies should lead to deeper understanding of the highly non-linear interaction between light and matter and better understanding of lasers in general. However, we have presented calculations using parameters for a real atomic system, but we do not claim to have that it is possible to construct a laser operating on the $4p_{3/2}$-$6p_{3/2}$ transition in potassium, since the value chosen for the cavity volume is quite optimistic with respect to present-day technology. An analysis of the linear stability together with many other aspects of the linear stability in two-photon processes such as transient behavior, instabilities *etc.*, will be discussed in another publication.

In conclusion, we have generalized the analysis of the single-mode homogeneously broadened single-photon laser in steady-state to the two-photon case. We have calculated the spatial behavior of the field strength and phase and have shown the effect of the additional non-linearity due to the two-photon coupling. We have also obtained the saturation behavior as a function of the parameters of the system. Although the model is rather idealized, its general features should be relevant to a real single-mode system.

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